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Existence and Uniqueness of Strong Solutions for Fractional Hybrid Integro-Differential Equations via Rothe's Method

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ABSTRACT

In this paper, we study a class of fractional hybrid integro-differential equations in a Banach algebra whose dual space is uniformly convex. The problem is formulated within an abstract framework involving a Riemann–Liouville fractional derivative, an unbounded linear operator generating an analytic semigroup, and a nonlinear Volterra-type integral term. By using Rothe's method, we establish the existence and uniqueness of a strong solution. An illustrative example is included to demonstrate the applicability of the theoretical results.

Keywords: Strong solution; Riemann-Liouville fractional derivative; Existence and uniqueness; Method of lines
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1 Introduction

Fractional calculus is an important branch of mathematics that generalizes the classical concepts of differentiation and integration to non-integer orders. Over the past few decades, it has become a powerful tool for modeling systems with memory and hereditary properties. Fractional order models provide a more accurate description of various physical and engineering phenomena, including viscoelasticity, anomalous diffusion, control theory, signal processing, and biological processes. Consequently, there has been significant interest in the theoretical study of fractional differential and integro-differential equations, particularly concerning the existence, uniqueness, and qualitative behavior of their solutions. See [1-6].

In parallel, considerable attention has been devoted to nonlinear differential equations with perturbations, commonly referred to as hybrid differential equations. A comprehensive overview of different kinds of perturbations affecting nonlinear differential and integral equations is provided by Dhage and Lakshmikantham [7]. Fundamental results concerning the existence of solutions to such hybrid equations have been established using hybrid fixed point theory. For further details, we refer to [8, 9] and the references cited therein.

Rothe's method, introduced in 1930 by Rothe [10], has played a significant role in the study of a wide range of mathematical problems and has been widely adopted for solving various types of differential equations. Its applications are particularly notable in the analysis of both integer and fractional order differential equations. Recognized as a reliable and practical tool, Rothe's method provides an effective numerical framework for approximating solutions to partial differential equations.

The fundamental idea of Rothe's method is to discretize the time domain using an appropriate scheme and evaluate the problem at uniformly spaced time levels. The solution over the entire time interval is then approximated through linear interpolation, and the resulting sequence of approximations is commonly

referred to as Rothe's functions. This approach transforms the original time-dependent problem into a sequence of simpler problems. By analyzing the limit of these approximations, the solution of the original problem can be obtained. Initially developed for the numerical treatment of parabolic differential equations in one spatial dimension [10], Rothe's method has since been extended to a broad class of mathematical models. Further details and applications of Rothe's method can be found in [11-13, 26].

In 2011, Ashyralyev [14] studied the following initial value problem,

$$\frac{dy}{dt} + D_{1/2}y(t) + Ay(t) = f(t), \text{ for } \quad 0 < t < 1,$$

$$y(0) = 0,$$

within a Banach space, where A is a strongly positive operator and $D_{\frac{1}{2}}$ denotes the Riemann–Liouville fractional derivative of order $\frac{1}{2}$. He established that this fractional initial value problem is well-posed. This type of fractional differential equation is associated with the classical Basset problem, which arises in fluid dynamics to describe the unsteady motion of a particle accelerating in a viscous fluid under gravitational force.

Several researchers have explored both analytical and numerical approaches for solving integro-differential equations involving fractional derivatives. In 2011, Awawdeh *et al.* [15] developed an analytical approach based on the homotopy analysis method to solve a class of fractional integro-differential equations. Subsequently, in 2013, Jingjun Zhao *et al.* [16] employed piecewise polynomial collocation techniques to address fractional integro-differential equations with weakly singular kernels. In 2016, Thabet and Dhakne [17] investigated the stability, existence, and uniqueness of solutions to fractional integro-differential equations involving the Caputo derivative in Banach spaces. Their analysis utilized the Darbo-Sadovskii fixed point theorem in conjunction with Hausdorff's measure of non-compactness. Inspired by the work of Ashyralyav [14], Bahuguna and Anjali [6] established the existence and uniqueness of strong solutions for a class of fractional differential equations in 2019. For more details, see [18-22].

Despite these developments, most existing works treat fractional differential equations, hybrid differential equations, and integro-differential equations separately, and primarily rely on fixed point method. In particular, the application of Rothe's time semi-discretization method to fractional hybrid integro-differential equations in Banach algebra have not been sufficiently explored.

The main objective of the present work is to address this gap by studying a class of fractional hybrid integro-differential equations in a Banach algebra whose dual space is uniformly convex. The problem involves a Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$, an unbounded linear operator generating an analytic semigroup of contractions, and a nonlinear Volterra-type integral term. By employing Rothe's method as an analytical tool, together with semigroup theory and appropriate estimates, we establish the existence and uniqueness of strong solutions under suitable Lipschitz-type conditions. This approach provides an alternative framework to the classical fixed point methods commonly used in the literature. It is worth emphasizing that the semi-discretization method is used purely for analytical purposes, and no numerical implementation is considered in this study.

Motivated by the above-mentioned literature, we consider the following fractional hybrid integro-differential equation

$$\frac{d}{dt}\left(\frac{u(t)}{f(t)}\right) + D^{\alpha}\left(\frac{u(t)}{f(t)}\right) + Au(t) = g(t) + \int_0^t p(t-x)k(x, u(x))dx,$$

$$u(0) = 0,$$

in a Banach algebra B, whose dual B^* is uniformly convex. The operator $-A$ is assumed to be the infinitesimal generator of an analytic semigroup of contractions on B. The derivative D^{α} denotes the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$. The function $f : [0, T] \to \mathbb{R} \setminus \{0\}$ is assumed to be continuous, while $g : [0, T] \to B$ is a continuous function. Furthermore, $p : [0, T] \to \mathbb{R}$ is a real-valued function, and $k : [0, T] \times B \to B$ is a nonlinear mapping.

The paper is organized as follows. In Section 2, we present the necessary definitions and preliminary results of fractional calculus. Section 3 focuses on developing the semi-discrete scheme and establishing essential estimates for the solution. In Section 4, we establish the main results concerning the existence and uniqueness of solutions. Finally, in Section 5, an illustrative example is provided to demonstrate the applicability of the theoretical results.

2 Assumptions and preliminaries

In this section, we review some key concepts, definitions, and fundamental results from fractional calculus. We also present several lemmas and hypotheses that will be used in the subsequent analysis.

Definition 2.1. [5] The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $g \in L^1((0,T);B)$ is given by the following definition

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(x)}{(t-x)^{1-\alpha}} dx, \ t > 0.$$

Definition 2.2. [5] The Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ for a function $g \in L^1((0,T);B)$ is defined as follows

$$D^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{g(x)}{(t-x)^{\alpha}} dx.$$

Lemma 2.3. *[6] Assume that the sequences x_n, y_n and z_n are nonnegative and*

$$x_n \leq y_n + \sum_{0 \leq k < n} z_k x_k \quad \text{for } n \geq 0,$$

then

$$x_n \leq y_n + \sum_{0 \leq k < n} y_k z_k exp\left(\sum_{k < j < n} z_j \right) \quad \text{for } n \geq 0.$$

Lemma 2.4. *[6] Suppose that $f(t)$ is a continuous, positive, and monotonically increasing function on the interval $[0,T]$ and let $y(t)$ be a non-negative function that is continuous (or piecewise continuous) on the interval $(0,T]$. Assume that there exist constants $A_1 > 0$, $A_2 > 0$ and α with $0 < \alpha < 1$ such that the inequality*

$$y(t) \leq A_1 f(t) + A_2 \int_0^t \frac{y(s)}{(t-s)^\alpha} ds, \ t \in [0,T],$$

holds. Thus, $y(t) \leq C f(t)$ for some constant C.

Lemma 2.5. *[23] We now define the sequence b_i by $b_i = (i+1)^{1-\alpha} - i^{1-\alpha}$, for $i = 1, 2, \ldots$*

1. *$b_i > b_{i+1}$, $i = 1, 2, \ldots$*
2. *$b_0 = 1$.*
3. *Let $0 < \alpha < 1$ and i be a nonnegative integer, then there exists a constant $C_\alpha > 0$, which depends only on α such that the inequality below holds,*

$$(i+1)^\alpha - i^\alpha \leq C_\alpha (i+1)^{\alpha-1},$$

where $C_\alpha = \max\left\{1, \alpha 2^{1-\alpha}\right\}$.

Lemma 2.6. *[24] If a C_0 semigroup of contractions has an infinitesimal generator, $-A$, then A is m-accretive; that is*
(i) $\langle Au, J(u) \rangle \geq 0, \quad \forall\, u \in D(A)$, where J denotes the dual map,
(ii) $R(I + \lambda A) = B$ for $\lambda > 0$.

Throughout the analysis, we assume that the following hypotheses hold.

(H1) Let $g : [0,T] \to B$ be a continuous function satisfying the Lipschitz condition,

$$\|g(t_1) - g(t_2)\| \leq L_g |t_1 - t_2|,$$

for all $t_1, t_2 \in [0,T]$ and $L_g \geq 0$ remains constant.

(H2) Let $p : [0,T] \to \mathbb{R}$ be a Lipschitz continuous function. Then there exists a constant $L_p \geq 0$ such that

$$|p(t_1) - p(t_2)| \leq L_p|t_1 - t_2|, \text{ for all } t_1, t_2 \in [0,T].$$

(H3) The operator $(I + A)^{-1}$ is compact.

(H4) Assume that $k : [0,T] \times B \to B$ is continuous and Lipschitz continuous with respect to both variables, i.e.,

$$\|k(t_1,u_1) - k(t_2,u_2)\| \leq L_k(|t_1 - t_2| + \|u_1 - u_2\|),$$

for all $t_1, t_2 \in [0,T]$ and $u_1, u_2 \in B$.

(H5) The function $f(t)$ is bounded on $[0,T]$ and $||f|| \leq M$, where M is a constant. Moreover, there exists a constant $m > 0$ such that

$$\inf_{t\in[0,T]} |f(t)| \geq m > 0.$$

(H6) $-A$ generates a C_0- semigroup of contractions.

Again, we consider our following fractional hybrid integro-differential equations,

$$\frac{d}{dt}\left(\frac{u(t)}{f(t)}\right) + D^{\alpha}\left(\frac{u(t)}{f(t)}\right) + Au(t) = g(t) + \int_0^t p(t-x)k(x,u(x))dx,\ 0 < t < T, \tag{1}$$

$$u(0) = 0, \tag{2}$$

in a Banach algebra B. We put

$$\frac{u(t)}{f(t)} = v(t),$$

or equivalently,

$$u(t) = f(t)v(t).$$

Consequently, equations (1) and (2) become:

$$\frac{d}{dt}\left(v(t)\right) + D^{\alpha}(v(t)) + Af(t)v(t) = g(t) + \int_0^t p(t-x)k(x,f(x)v(x))dx,\ 0 < t < T, \tag{3}$$

$$v(0) = 0, \tag{4}$$

in a Banach algebra B.

3 Discretization scheme and a-priori estimates

In this section, we construct a time semi-discrete scheme for the given problem (1)-(2) using Rothe's method and derive the necessary a priori estimates for the approximate solutions.

3.1 Time Discretization

We divide the interval $[0,T]$ into n equal subintervals of length $h_n = \frac{T}{n}$. Define the discrete time levels by

$$t_i^n = ih_n, \quad i = 1,2,\ldots,N.$$

Let v_i^n denote the approximation of the solution $v(t)$ at time t_i^n.

3.2 L1 Method for the Fractional Derivative

At each time $t = t_i^n$, the Riemann–Liouville fractional derivative is approximated (see [25]) as

$$D^{\alpha}v(t_i^n) \approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=1}^{i} b_{i-j}\frac{v_j^n - v_{j-1}^n}{h_n}h_n^{1-\alpha} \tag{5}$$

$$= \sum_{j=1}^{i}(v_j^n - v_{j-1}^n)d_j^{i,n},$$

where $b_l = (l+1)^{1-\alpha} - l^{1-\alpha}$ and $d_j^{i,n} = b_{i-j}\dfrac{h_n^{-\alpha}}{\Gamma(2-\alpha)}$.

3.3 Construction of the Semi-Discrete Scheme

We now construct the semi-discrete version of the problem (3)–(4). In $t = t_j^n$, we substitute Eqs. (3)–(4) with the following approximate equations:

$$\frac{v_j^n - v_{j-1}^n}{h_n} + Af_j^n v_j^n + D^\alpha v_j^n = g_j^n + h_n \sum_{i=0}^{j-1} p_{ji}^n k_i^n \tag{6}$$

$$v_0^n = 0, \tag{7}$$

where, $g_j^n = g(t_j^n)$, $p_{ji}^n = p(t_j^n - t_i^n)$, $f_j^n = f(t_j^n)$ and $k_i^n = k(t_i^n, f_i^n v_i^n)$.

The m-accretivity of A ensures the existence and uniqueness of $v_j^n \in D(A)$ satisfying equations (6)–(7).

3.4 A Priori Estimates

Before establishing the main result, we derive some preliminary results required for the proof.

Lemma 3.1. *Suppose that the assumptions (**H1**)-(**H6**) hold. Then there exists a positive constant C, independent of j, h and n such that*

$$\|v_j^n\| \le C, \text{ for } \quad j = 1, 2, \ldots, n.$$

Proof. Taking $j = 1$ in equation (6), we get

$$\frac{v_1^n - v_0^n}{h_n} + Af_1^n v_1^n + D^\alpha v_1^n = g_1^n + h_n p_{10}^n k_0^n,$$

or

$$\frac{v_1^n - v_0^n}{h_n} + Af_1^n v_1^n + \frac{1}{\Gamma(2-\alpha)} \frac{v_1^n - v_0^n}{h_n} h_n^{1-\alpha} = g_1^n + h_n p_{10}^n k_0^n.$$

Rearranging the above equation, we obtain

$$\left(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right) v_1^n + h_n Af_1^n v_1^n = h_n g_1^n + h_n^2 p_{10}^n k_0^n.$$

Using the accretivity of A, we obtain

$$\left(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right) \|v_1^n\| \le h_n \|g_1^n\| + h_n^2 \|p_{10}^n k_0^n\|.$$

Since $\left(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right) > 1$, it follows that

$$\|v_1^n\| \le h_n \|g_1^n\| + h_n^2 \|p_{10}^n k_0^n\|.$$

For $j \ge 2$, applying (5) in (6), we obtain

$$\begin{aligned}
&\frac{v_j^n - v_{j-1}^n}{h_n} + Af_j^n v_j^n + D^\alpha v_j^n = g_j^n + h_n \sum_{i=0}^{j-1} p_{ji}^n k_i^n \\
\implies & v_j^n + h_n Af_j^n v_j^n + h_n d_j^{j,n} v_j^n = v_{j-1}^n + h_n g_j^n + h_n \sum_{i=1}^{j-1} (d_{i+1}^{j,n} - d_i^{j,n}) v_i^n + h_n^2 \sum_{i=0}^{j-1} p_{ji}^n k_i^n \\
\implies & \left(1 + h_n d_j^{j,n}\right) v_j^n + h_n Af_j^n v_j^n = v_{j-1}^n + h_n g_j^n + h_n \sum_{i=1}^{j-1} \left(d_{i+1}^{j,n} - d_i^{j,n}\right) v_i^n + h_n^2 \sum_{i=0}^{j-1} p_{ji}^n k_i^n.
\end{aligned}$$

Using the accretivity of A and the inequality $\|v_j^n\| \le (1 + h_n d_j^{j,n}) \|v_j^n\|$, we obtain

$$\begin{aligned}
\|v_j^n\| &\le \|v_{j-1}^n\| + h_n \|g_j^n\| + h_n \sum_{i=1}^{j-1} \left(d_{i+1}^{j,n} - d_i^{j,n}\right) \|v_i^n\| + h_n^2 \sum_{i=0}^{j-1} \|p_{ji}^n\| \|k_i^n\| \\
&\le \|v_{j-1}^n\| + h_n \|g_j^n\| + h_n \sum_{i=1}^{j-1} \left(d_{i+1}^{j,n} - d_i^{j,n}\right) \|v_i^n\| + p_0 h_n^2 \sum_{i=0}^{j-1} \|k_i^n\|.
\end{aligned}$$

Let $p_0 = \max|p(t)|$. By hypothesis (H4), we have

$$\begin{aligned}\|v_j^n\| &\leq \|v_{j-1}^n\| + h_n\|g_j^n\| + h_n\sum_{i=1}^{j-1}\left(d_{i+1}^{j,n} - d_i^{j,n}\right)\|v_i^n\| \\ &\quad + p_0h_n\left(\|k_0\|T + L_kT^2 + L_kh_nM\left(\|v_1^n\| + \cdots + \|v_{j-1}^n\|\right)\right) \\ &\leq h_n(||g_j^n|| + p_0||k_0|| + L_kT^2) + \left(1 + L_kp_0h_n^2M + h_n\left(d_j^{j,n} - d_{j-1}^{j,n}\right)\right)||v_{j-1}^n|| \\ &\quad + h_n\sum_{i=1}^{j-2}\left(d_{i+1}^{j,n} - d_i^{j,n} + L_kp_0h_nM\right)||v_i^n||. \end{aligned} \tag{8}$$

Define

$$G_j = h_n(\|g_j^n\| + p_0\|k_0\| + L_kT^2), \quad P_{j-1} = 1 + L_kp_0h_n^2M + h_n(d_j^{j,n} - d_{j-1}^{j,n})$$

and

$$P_i = h_n(d_{i+1}^{j,n} - d_i^{j,n}) + L_kp_0h_n^2M, \quad \text{for } i = 1, \ldots, j-2.$$

Then,

$$\begin{aligned}\sum_{1\leq i<j} P_i &= \sum_{i=1}^{j-1} h_n\left(d_{i+1}^{j,n} - d_i^{j,n}\right) + (j-1)L_kp_0h_n^2M + 1 \\ &= h_n\left(d_j^{j,n} - d_1^{j,n}\right) + (j-1)L_kp_0h_n^2M + 1.\end{aligned}$$

Now, applying Lemma 2.3 in (8), we conclude that there exists a constant $C > 0$ such that

$$||v_j^n|| \leq C.$$

■

We now define

$$\delta v_j^n = \frac{v_j^n - v_{j-1}^n}{h_n}, \quad \text{for } j = 1, 2, \ldots, n.$$

Lemma 3.2. *Assuming that hypotheses (**H1**)-(**H6**) hold. Then there exists a constant $C_1 > 0$, independent of j, h and n, such that*

$$\|\delta v_j^n\| \leq C_1, \quad j = 1, 2, \ldots, n.$$

Proof. For $j = 1$ in equation (6), we obtain

$$\frac{v_1^n - v_0^n}{h_n} + Af_1^nv_1^n + \frac{1}{\Gamma(2-\alpha)}\frac{v_1^n - v_0^n}{h_n^\alpha} = g_1^n + h_np_{10}^nk_0^n,$$

i.e.,

$$\left(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\frac{v_1^n - v_0^n}{h_n} + Af_1^nv_1^n = g_1^n + h_np_{10}^nk_0^n.$$

Using the accretivity of A, we obtain

$$\|\delta v_1^n\| \leq \|g_1^n\| + h_np_0\|k_0\|.$$

For $j \geq 2$, subtracting equation (6) at $j-1$ from equation (6) at j, we obtain:

$$\begin{aligned}\frac{v_j^n - v_{j-1}^n}{h_n} + Af_j^nv_j^n - Af_{j-1}^nv_{j-1}^n + D^\alpha v_j^n - D^\alpha v_{j-1}^n &= \frac{v_{j-1}^n - v_{j-2}^n}{h_n} + g_j^n - g_{j-1}^n \\ &\quad + h_n\sum_{i=0}^{j-1}p_{ji}^nk_i^n - h_n\sum_{i=0}^{j-2}p_{j-1i}^nk_i^n. \end{aligned} \tag{9}$$

Substituting equation (5) into equation (9), we get

$$\begin{aligned}\frac{v_j^n - v_{j-1}^n}{h_n} + Af_j^n v_j^n - Af_{j-1}^n v_{j-1}^n + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\delta v_j^n &= \delta v_{j-1}^n + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-1} b_{j-1-i}\delta v_i^n h_n^{1-\alpha} \\ &\quad - \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-1} b_{j-i}\delta v_i^n h_n^{1-\alpha} + g_j^n - g_{j-1}^n \\ &\quad + h_n p_{jj-1}^n K_{j-1}^n + h_n \sum_{i=0}^{j-2}(p_{ji}^n - p_{j-1i}^n)k_i^n.\end{aligned}$$

Hence,

$$\begin{aligned}\left(1 + \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\delta v_j^n + Af_j^n v_j^n - Af_{j-1}^n v_{j-1}^n &= \left(1 + (b_0 - b_1)\frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\delta v_{j-1}^n + g_j^n - g_{j-1}^n \\ &\quad + h_n p_{jj-1}^n k_{j-1}^n + h_n \sum_{i=0}^{j-2}(p_{ji}^n - p_{j-1i}^n)k_i^n \\ &\quad + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-2}(b_{j-1-i} - b_{j-i})\delta v_i^n h_n^{1-\alpha}.\end{aligned}$$

Using the accretivity of A, together with hypotheses (H1)-(H6) and Lemma 3.1, we obtain

$$\begin{aligned}\|\delta v_j^n\| &\le \left(1 + (b_0 - b_1)\frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\|\delta v_{j-1}^n\| + \|g_j^n - g_{j-1}^n\| + h_n p_0 \|k_{j-1}^n\| \\ &\quad + h_n \sum_{i=0}^{j-2}\left(|p_{ji}^n - p_{j-1i}^n|\right)\|k_i^n\| + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-2}(b_{j-1-i} - b_{j-i})\|\delta v_i^n\| h_n^{1-\alpha} \\ &\le \left(1 + (b_0 - b_1)\frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right)\|\delta v_{j-1}^n\| + L_g h_n \\ &\quad + \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j-2}(b_{j-1-i} - b_{j-i})\|\delta v_i^n\| h_n^{1-\alpha} + h_n C.\end{aligned}$$

Therefore, by applying Lemma 2.3, we conclude that there exists a constant $C_1 > 0$ such that

$$\|\delta v_j^n\| \le C_1.$$

■

Lemma 3.3. *Assume that hypotheses (H1)-(H6) are satisfied. Then there exist constants $C_2 > 0$ and $C_3 > 0$, independent of j, h and n, such that*

$$\|D^\alpha v_j^n\| \le C_2 \quad \text{and} \quad \|Af_j^n v_j^n\| \le C_3, \quad \text{for } j = 1, 2, \ldots, n.$$

Proof. We first recall that

$$D^\alpha v_j^n = \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{j} b_{j-i}\delta v_i^n h_n^{1-\alpha}.$$

Using Lemma 2.5 together with Lemma 3.2, we obtain

$$\begin{aligned}\|D^\alpha v_j^n\| &\le \frac{C_\alpha C_1}{\Gamma(2-\alpha)}\sum_{i=1}^{j}\frac{h_n}{[(j-i+1)h_n]^\alpha} \\ &\le \frac{C_\alpha C_1}{\Gamma(2-\alpha)}\sum_{i=1}^{j}\int_{t_{i-1}^n}^{t_i^n}\frac{ds}{(t_j^n - s)^\alpha} \\ &\le \frac{C_\alpha C_1}{\Gamma(2-\alpha)}\int_0^{t_j^n}\frac{ds}{(t_j^n - s)^\alpha} \\ &\le \frac{C_\alpha C_1 T^{1-\alpha}}{(1-\alpha)\Gamma(2-\alpha)} = C_2.\end{aligned}$$

Next, applying the triangle inequality together with equation (6), Lemma 3.1, Lemma 3.2, and assumption (H4), we obtain

$$\|Af_j^n v_j^n\| \leq \|g_j^n\| + \|\delta v_j^n\| + \|D^\alpha v_j^n\| + h_n p_0 \sum_{i=1}^{j} \|k_i^n\| \leq C_3.$$

■

We now introduce a sequence of functions $Y^n : [0,T] \to D(A)$ defined by

$$Y^n(t) = \begin{cases} 0, & t = 0, \\ v_j^n, & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

Furthermore, we define Rothe's sequence $Z^n : [0,T] \to D(A)$ by

$$Z^n(t) = \begin{cases} v_0, & t = 0, \\ v_{j-1}^n + \frac{t - t_{j-1}^n}{h_n}\left(v_j^n - v_{j-1}^n\right), & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

Remark 1. From Lemma 3.1 and Lemma 3.2, it follows that the functions $Z^n(t)$ is uniformly Lipschitz continuous on $[0,T]$. Moreover, as $n \to \infty$, we have

$$Z^n(t) - Y^n(t) \to 0 \quad \text{in} \quad B, \quad \text{for} \quad t \in (0,T].$$

In addition, both sequences $Z^n(t)$ and $Y^n(t)$ are uniformly bounded in B.

Next, we introduce a sequence of step functions $\bar{D}^\alpha Z^n : [0,T] \to B$ defined by

$$\bar{D}^\alpha Z^n(t) = \begin{cases} 0, & t = 0, \\ \left(\sum_{i=1}^{j} b_{j-i} \frac{(v_i^n - v_{i-1}^n)}{h_n} \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)}\right), & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

Lemma 3.4. *As $n \to \infty$, $\|\bar{D}^\alpha Z^n(t) - D^\alpha Z^n(t)\| \to 0$ uniformly on $(0,T]$.*

Proof. Given any t, there exists a j such that $t \in (t_{j-1}^n, t_j^n]$. Therefore

$$\begin{aligned}
&\left\|\bar{D}^\alpha Z^n(t) - D^\alpha Z^n(t)\right\| \\
&= \| \sum_{i=1}^{j} b_{j-i} \frac{v_i^n - v_{i-1}^n}{h_n} \frac{h_n^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{j-1} \frac{v_i^n - v_{i-1}^n}{h_n} \left[(t - t_{i-1}^n)^{1-\alpha} - (t - t_i^n)^{1-\alpha}\right] \\
&\quad - \frac{v_j^n - v_{j-1}^n}{h_n} \frac{(t - t_{j-1}^n)^{1-\alpha}}{\Gamma(2-\alpha)} \| \\
&\leq \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{j-1} \left\| \frac{v_i^n - v_{i-1}^n}{h_n} \right\| \left[(t - t_{i-1}^n)^{1-\alpha} - (t - t_i^n)^{1-\alpha} - ((j-i+1)h_n)^{1-\alpha} + ((j-i)h_n)^{1-\alpha}\right] \\
&\quad + \frac{1}{\Gamma(2-\alpha)} \left\| \frac{v_j^n - v_{j-1}^n}{h_n} \right\| \left[h_n^{1-\alpha} - (t - t_{j-1}^n)^{1-\alpha}\right] \\
&\leq \frac{C}{\Gamma(2-\alpha)} \left[t^{1-\alpha} - (t - t_{j-1}^n)^{1-\alpha} - (t_j^n)^{1-\alpha} - h_n^{1-\alpha}\right] \\
&\quad + \frac{C}{\Gamma(2-\alpha)} \left[(h_n)^{1-\alpha} - (t - t_{j-1}^n)^{1-\alpha}\right] \\
&\leq \frac{2C}{\Gamma(2-\alpha)} \left[h_n^{1-\alpha} - (t - t_{j-1}^n)^{1-\alpha}\right] + \frac{C}{\Gamma(2-\alpha)} \left[(t_j^n)^{1-\alpha} - t^{1-\alpha}\right] \\
&\leq \frac{2C}{\Gamma(2-\alpha)} h_n^{1-\alpha} + \frac{C}{\Gamma(2-\alpha)} \left[(t_j^n) - t\right]^{1-\alpha} \\
&\leq \frac{3C}{\Gamma(2-\alpha)} h_n^{1-\alpha} \to 0 \quad \text{as} n \to \infty.
\end{aligned}$$

■

Lemma 3.5. *A subsequence $\{Z^{n_k}\}$ of $\{Z^n\}$ exists, along with a function $v : [0,T] \to D(A)$ such that $Z^{n_k} \to v$ in $C([0,T],B)$ and $AY^{n_k}(t) \to Av(t)$ uniformly in B as $n \to \infty$. Moreover, $Av(t)$ is weakly continuous.*

Proof. The proof follows as a consequence of Theorem 2.1 and Lemma 2.3 in [26]. ■

Lemma 3.6. *There exists a subsequence $\{Z^{n_k}\}$ of $\{Z^n\}$ such that, as $n \to \infty$,*

$$\frac{d^-}{dt} Z^{n_k} \rightharpoonup \frac{dv}{dt} \quad \text{and} \quad \bar{D}^\alpha Z^{n_k} \rightharpoonup D^\alpha v \quad \text{in} \quad L^2([0,T],B).$$

Proof. Since $D^\alpha Z^n(t)$ is uniformly bounded in $L^2([0,T],B)$, there exists a subsequence $D^\alpha Z^{n_k}(t)$ and a function $\psi \in L^2([0,T],B)$ such that

$$D^\alpha Z^{n_k} \rightharpoonup \psi \text{ in } L^2([0,T],B).$$

Define $\phi^n : [0,T] \to B$ by

$$\phi^n(t) = I^{1-\alpha} Z^n(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{Z^n(s)}{(t-s)^\alpha} ds.$$

For $t \in (0, t_1^n)$, we have

$$Z^n(t) = v_0^n + \frac{(t-t_0^n)}{h_n}(v_1^n - v_0^n) = \frac{t}{h_n} v_1^n.$$

Therefore,

$$\phi^n(t) = \frac{1}{\Gamma(1-\alpha)} \frac{v_1^n}{h_n} \int_0^t \frac{s}{(t-s)^\alpha}\, ds = \frac{1}{\Gamma(3-\alpha)} \delta v_1^n t^{2-\alpha}.$$

Since δv_1^n is bounded, it follows that $\phi^n(0_+) = 0$ for all n.

For $b^* \in B^*$, we observe that

$$\langle \phi^{n_k}(t), b^* \rangle = \int_0^t \langle D^\alpha Z^{n_k}(s), b^* \rangle ds.$$

As $Z^{n_k} \to v$ in $C([0,T],B)$, it follows that

$$\phi^{n_k}(t) \to \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v(s)}{(t-s)^\alpha} ds, \quad \text{as} \quad k \to \infty.$$

Taking the limit as $k \to \infty$, we obtain

$$\left\langle \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v(s)}{(t-s)^\alpha} ds, b^* \right\rangle = \int_0^t \langle \psi(s), b^* \rangle\, ds.$$

Therefore

$$\psi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(s)}{(t-s)^\alpha} ds.$$

Analogously, it follows that $\frac{d^-}{dt} Z^{n_k} \rightharpoonup \frac{dv}{dt}$ in $L^2([0,T],B)$. ■

Remark 2. As $n \to \infty$, $\bar{D}^\alpha Z^n \rightharpoonup D^\alpha v$ in $L^2([0,T],B)$.

We now introduce the sequence of functions $K^n(t)$ defined by

$$K^n(t) = \begin{cases} h_n p_{10} k_0, & t = 0, \\ h_n \sum_{i=0}^{j-1} p_{ji} k_i, & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

Lemma 3.7. *For a piecewise continuous function $y(t)$, we define*

$$K(y)(t) = \int_0^t p(t-s) k(s, f(s) y(s)) ds,$$

Then, as $n \to \infty$, $K^n(t)$ converges uniformly to $K(v)(t)$ on the interval $[0,T]$.

Proof. The result follows from Lemma 2.4 in [26]. ■

Lemma 3.8. *In* $[0,T]$, $Af(t)v(t)$ *is Bochner integrable.*

Proof. The proof follows from Theorem 2.1 in [27]. ■

Next, we define the step functions $g^n(t)$ and $f^n(t)$ as follows

$$g^n(t) = \begin{cases} g(0), & t = 0, \\ g(t_j^n), & t \in (t_{j-1}^n, t_j^n], \end{cases}$$

and

$$f^n(t) = \begin{cases} f(0), & t = 0, \\ f(t_j^n), & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

4 Main results

The following theorem will give the existence and uniqueness of strong solution of the differential equation (1)-(2).

Theorem 4.1. *Suppose that* $-A$ *is the infinitesimal generator of an analytic semigroup of contractions and the assumptions (H1) - (H6) are satisfied. Then, the fractional hybrid integro-differential equation* (1)-(2) *has a unique strong solution.*

Proof. **Existence:** First, we recall the notation $Z^n(t)$, $f^n(t)$, $Y^n(t)$, $g^n(t)$, and $K^n(t)$. Then, equation (6) can be rewritten as

$$\frac{d}{dt}Z^n(t) + Af^n(t)Y^n(t) + \bar{D}^\alpha Z^n(t) = g^n(t) + K^n(t),$$

where $\frac{d}{dt}$ denotes the left derivative in the interval $(0,T]$. For any $t \in (0,T]$, we obtain

$$\int_0^t Af^n(s)Y^n(s)\,ds = -Z^n(t) + \int_0^t g^n(s)\,ds + \int_0^t K^n(s)\,ds - \int_0^t \bar{D}^\alpha Z^n(s)\,ds. \tag{10}$$

Since B^* be the dual space of B. For every $b^* \in B^*$, we have

$$\int_0^t \langle Af^n(s)Y^n(s), b^*\rangle\,ds = -\langle Z^n(t), b^*\rangle + \int_0^t \langle g^n(s) + K^n(s), b^*\rangle\,ds - \int_0^t \langle \bar{D}^\alpha Z^n(s), b^*\rangle\,ds.$$

Now consider a subsequence n_k of n. Then

$$\int_0^t \langle Af^{n_k}(s)Y^{n_k}(s), b^*\rangle\,ds = -\langle Z^{n_k}(t), b^*\rangle + \int_0^t \langle g^{n_k}(s) + K^{n_k}(s), b^*\rangle\,ds - \int_0^t \langle \bar{D}^\alpha Z^{n_k}(s), b^*\rangle\,ds.$$

We recall the Bounded Convergence Theorem for Banach space-valued functions, which states that if $\{f_n\}$ converges almost everywhere to f and is uniformly bounded on a set of finite measure, then

$$\int_X f_n\,d\mu \to \int_X f\,d\mu \quad \text{in } B.$$

Applying this theorem together with Lemma 3.5, Lemma 3.7, and Remark 2, we obtain, as $k \to \infty$,

$$\int_0^t \langle Af(s)v(s), b^*\rangle\,ds = -\langle v(t), b^*\rangle + \int_0^t \langle g(s) + Kv(s), b^*\rangle\,ds - \int_0^t \langle D^\alpha v(s), b^*\rangle\,ds. \tag{11}$$

By Lemma 3.5, $Z^{n_k}(t) \to v(t)$ in $C([0,T],B)$ and $AY^{n_k}(t) \to Av(t)$ uniformly on $[0,T]$. Moreover, Lemma 3.7 ensures that $K^{n_k}(t)$ converges uniformly to $K(v)(t)$ on $[0,T]$, while Remark 2 yields $\bar{D}^\alpha Z^{n_k} \rightharpoonup D^\alpha v$ in $L^2([0,T],B)$.

Since

$$\int_0^t \langle D^\alpha v(s), b^*\rangle ds = \langle I^{1-\alpha} v(t), b^*\rangle.$$

Equation (11) can be rewritten as

$$\langle v(t) + I^{1-\alpha} v(t), b^*\rangle = -\int_0^t \langle Af(s)v(s), b^*\rangle ds + \int_0^t \langle g(s) + K(v)(s), b^*\rangle ds.$$

Since the integrands are continuous and uniformly bounded on the right-hand side for each $b^* \in B^*$, it follows that $\langle v(t) + I^{1-\alpha} v(t), b^*\rangle$ is continuously differentiable. Moreover, $v(t) + I^{1-\alpha} v(t)$ possesses a strong derivative almost everywhere on $[0,T]$, since $Af(t)v(t)$ is Bochner integrable. Consequently,

$$\frac{d}{dt}\left(v(t) + I^{1-\alpha} v(t)\right) = -Af(t)v(t) + g(t) + \int_0^t p(t-s)k(s, f(s)v(s))\, ds, \quad \text{a.e. on } [0,T].$$

Since $v(t)$ is continuous, the fractional integral $I^{1-\alpha} v(t)$ is differentiable [28], which implies that $v(t)$ is differentiable. Therefore,

$$\frac{dv}{dt} + D^\alpha v(t) + Af(t)v(t) = g(t) + \int_0^t p(t-s)k(s, f(s)v(s))\, ds, \quad \text{a.e. } t \in [0,T].$$

Thus, $v(t)$ is a strong solution of the problem (3)-(4).

Uniqueness: Let v_1 and v_2 be two strong solutions of the problem (3)-(4). Then

$$\frac{dv_1}{dt} + D^\alpha v_1(t) + Af(t)v_1(t) = g(t) + \int_0^t p(t-s)k(s, f(s)v_1(s))\, ds, \quad \text{a.e. } t \in [0,T].$$

and

$$\frac{dv_2}{dt} + D^\alpha v_2(t) + Af(t)v_2(t) = g(t) + \int_0^t p(t-s)k(s, f(s)v_2(s))\, ds, \quad \text{a.e. } t \in [0,T].$$

Put $w = v_1 - v_2$. Then,

$$\frac{dw}{dt} + D^\alpha w(t) + Af(t)w(t) = \int_0^t p(t-s)\big(k(s, f(s)v_1(s)) - k(s, f(s)v_2(s))\big) ds.$$

Define

$$G(t) = \int_0^t p(t-s)\left(k(s, f(s)v_1(s)) - k(s, f(s)v_2(s))\right) ds.$$

Let $T(t)$ be the semigroup generated by $-A$. Since $-A$ generates a strongly continuous semigroup and the associated terms are Bochner integrable, we obtain

$$w(t) = -\int_0^t T(t-s)D^\alpha w(s) ds + \int_0^t T(t-s)G(s) ds. \tag{12}$$

As $w(t)$ is differentiable almost everywhere and $-A$ be the infinitesimal generator of an analytic semigroup, differentiating (12) with respect to t, and using the property $\frac{d}{dt}T(t) = AT(t)$, we get

$$\begin{aligned} w'(t) = -D^\alpha w(t) + G(t) + \int_0^t AT(t-s)D^\alpha w(s)\, ds \\ - \int_0^t AT(t-s)G(s)\, ds. \end{aligned} \tag{13}$$

By definition 2.2,

$$D^\alpha w(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{w(p)}{(t-p)^\alpha} dp = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w'(p)}{(t-p)^\alpha} dp. \tag{14}$$

Substituting Equation (13) into Equation (14), we obtain

$$\begin{aligned} D^{\alpha}w(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{(-D^{\alpha}w(s)+G(s))}{(t-s)^{\alpha}}ds \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_s^t \frac{1}{(t-p)^{\alpha}}AT(p-s)D^{\alpha}w(s)\,dp\,ds \\ &- \frac{1}{\Gamma(1-\alpha)}\int_0^t\int_s^t \frac{1}{(t-p)^{\alpha}}AT(p-s)G(s)\,dp\,ds. \end{aligned} \tag{15}$$

By Theorem 2.1 in [14], there exists a constant $M_1 > 0$ such that

$$\left\| \frac{1}{\Gamma(1-\alpha)}\int_s^t \frac{1}{(t-p)^{\alpha}}AT(p-s)\,dp \right\| \leq \frac{M_1}{(t-s)^{\alpha}}. \tag{16}$$

Using (15) and (16), we obtain

$$\|D^{\alpha}w(t)\| \leq M_1\int_0^t \frac{\|D^{\alpha}w(s)\|}{(t-s)^{\alpha}}ds + M_1\int_0^t \frac{\|G(s)\|}{(t-s)^{\alpha}}ds.$$

Next, we estimate $\|G(t)\|$ using assumption (H4),

$$\begin{aligned} \|G(t)\| &\leq p_0\int_0^t \|k(s,f(s)v_1(s)) - k(s,f(s)v_2(s))\|\,ds \\ &\leq L_k p_0 M\int_0^t \|v_1(s)-v_2(s)\|\,ds \\ &\leq L_k p_0 MT\|w\|_t, \quad p_0 = \max|p(t)|, \end{aligned}$$

in which $\|w\|_t = \sup_{0\leq s\leq t}\|w(s)\|$. Thus,

$$\|D^{\alpha}w(t)\| \leq A_1\|w\|_t + M_1\int_0^t \frac{\|D^{\alpha}w(s)\|}{(t-s)^{\alpha}}ds, \quad \text{where } A_1 = \frac{M_1L_kp_0MTt^{1-\alpha}}{1-\alpha}.$$

By Lemma 2.4, it follows that

$$\|D^{\alpha}w(t)\| \leq C\|w\|_t.$$

Using the accretivity of A, we obtain

$$\begin{aligned} \left\langle \frac{dw(t)}{dt}, Jw(t)\right\rangle &= \langle G(t) - D^{\alpha}w(t) - Af(t)w(t), Jw(t)\rangle \\ &\leq \langle G(t) - D^{\alpha}w(t), Jw(t)\rangle \\ &\leq (\|G(t)\| + \|D^{\alpha}w(t)\|)\|w(t)\| \\ &\leq M_1\|w\|_t^2. \end{aligned} \tag{17}$$

Since

$$\frac{d}{dt}\|w(t)\|^2 = 2\left\langle \frac{dw(t)}{dt}, Jw(t)\right\rangle.$$

Now integrating equation (17) over $[0,t]$, we get

$$\|w(t)\|^2 \leq 2M_1\int_0^t \|w\|_s^2 ds.$$

Therefore

$$\|w\|_t^2 \leq 2M_1\int_0^t \|w\|_s^2 ds.$$

Applying Gronwall's inequality, we conclude that

$$\|w\|_t = 0 \quad \text{for all } t\in[0,T].$$

Therefore, $w(t) = 0$ i.e. $v_1(t) - v_2(t) = 0 \implies v_1(t) = v_2(t)$. This proves the uniqueness of the strong solution of the problem (3)-(4). This completes the proof of the theorem. ■

5 Example

We consider the following fractional hybrid integro-differential equation

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{u(t,y)}{1+e^{-t}}\right)+D^{\alpha}\left(\frac{u(t,y)}{1+e^{-t}}\right)-\frac{\partial^2 u(t,y)}{\partial y^2}\\ =t\sin(\pi y)+\int_0^t \mathrm{e}^{-(t-s)}\tanh(u(s,y))\,ds,\\ 0\le y\le 1,\ 0<t\le T,\end{aligned}\tag{18}$$

along with the following boundary and initial conditions

$$u(t,0)=u(t,1)=0,\tag{19}$$

and

$$u(0,y)=0,\quad y\in[0,1].\tag{20}$$

Here, $\alpha\in(0,1)$ and D^{α} denotes the Riemann-Liouville fractional derivative of order α.

Let $B=L^2(0,1)$ and define the operator A by $Au=-u''$ with domain $D(A)=\{u(\cdot)\in B: u,u' \text{ are absolutely continuous, } u''\in B \text{ and } u(0)=u(1)=0\}$. The operator $-A$ generates a compact analytic semigroup of contractions. Therefore, the resolvent operator $(I+A)^{-1}$ is compact[24].

Let us consider the following functions:

$$g(t,y)=t\sin(\pi y),\quad p(t)=e^{-t},\quad k(t,u)=\tanh(u),\quad f(t)=1+e^{-t}.$$

We now proceed to define the associated functions

$$u:[0,T]\to B,\quad g:[0,T]\to B,\quad k:[0,T]\times B\to B,$$

s.t.

$$u(t)(y)=u(t,\,y),\ g(t)(y)=g(t,\,y),\ \text{ and }\ k(t,\,u(.))(y)=k(t,\,u(t,y)).$$

Similarly, $f:[0,T]\to\mathbb{R}\setminus 0$ and $p:[0,T]\to\mathbb{R}$.

Since

$$g(t)(y)=t\sin(\pi y),$$

for each $t\in[0,T]$,

$$\int_0^1 |g(t,y)|^2\,dy=t^2\int_0^1\sin^2(\pi y)\,dy=\frac{t^2}{2}<\infty.$$

Hence, $g(t,\cdot)\in L^2(0,1)$. Thus, $g:[0,T]\to B$ is well-defined and continuous. Moreover, $g(t,y)$ is Lipschitz continuous in t for each $y\in[0,1]$, since

$$|g(t_1,y)-g(t_2,y)|\le|t_1-t_2|.$$

The function $p(t)=e^{-t}$ is continuous and satisfies

$$|p(t_1)-p(t_2)|\le|t_1-t_2|.$$

Further, the function $k(t,u)=\tanh(u)$ is continuous. For any $u\in B$, we have

$$|k(t,u(y))|=|\tanh(u(y))|\le 1,$$

which implies

$$\int_0^1|k(t,u(y))|^2dy\le 1.$$

Hence, $k(t,u(\cdot))\in L^2(0,1)$, and therefore the induced mapping

$$k:[0,T]\times B\to B$$

defined by

$$k(t,u)(y) = \tanh(u(y))$$

is well-defined. Moreover,

$$|k(t_1,u_1) - k(t_2,u_2)| = |\tanh(u_1) - \tanh(u_2)| \leq |t_1 - t_2| + |u_1 - u_2|.$$

We have

$$f(t) = 1 + e^{-t},\ t \in [0,T].$$

Now, at $t = 0$,

$$f(0) = 1 + e^0 = 2,$$

bounds on $[0,T]$. Since, $e^{-t} \in [e^{-T}, 1]$, we obtain

$$\begin{aligned} & 1 + e^{-T} \leq f(t) \leq 2 \\ \implies & |f(t)| \leq 2. \end{aligned}$$

Now,

$$\inf_{t\in[0,T]} |f(t)| = 1 + e^{-T} > 0.$$

Thus, there exists a constant $m > 0$ such that

$$\inf_{t\in[0,T]} |f(t)| \geq m > 0,$$

with $m = 1 + e^{-T}$.

Therefore, all the hypothesis (H1)–(H6) are satisfied. Consequently, equations (18)-(20) can be reformulated as

$$\frac{d}{dt}\left(\frac{u(t)}{f(t)}\right) + D^{\alpha}\left(\frac{u(t)}{f(t)}\right) + Au(t) = g(t) + \int_0^t p(t-s)k(s,u(s))\,ds,$$

$$u(0) = 0.$$

Hence, all the assumptions of Theorem 4.1 are satisfied. Therefore, the problem defined by (18)-(20) admits a unique strong solution. This confirms the applicability of the theoretical results to a concrete class of fractional partial differential equations.

The above example demonstrates that the abstract results obtained in this work are applicable to a class of fractional partial differential equations involving memory effects and nonlinear perturbations. Such models naturally arise in the study of heat conduction with memory, viscoelastic materials, and anomalous diffusion processes.

In particular, the presence of the fractional derivative captures hereditary properties, while the integral term represents accumulated past effects. The presence of the nonlinear function further captures realistic interactions within the system. Therefore, this example highlights the practical relevance of the theoretical results and shows that the developed framework is not limited to purely abstract settings.

6 Conclusion

In this work, we studied a class of fractional hybrid integro-differential equations in a Banach algebra. By using Rothe's method, we established the existence and uniqueness of a strong solution under suitable assumptions. An example was also provided to illustrate the applicability of the obtained theoretical results.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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