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Implicit Hybrid Iterations

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ABSTRACT

In this paper, implicit hybrid iteration is defined, which is hybrid of Picard iteration with implicit Mann and Ishikawa iterations. Strong convergence of these iterations are discussed. By numerical example, it is proved that the newly introduced iteration has better convergence rate than well known implicit Mann iteration as well as implicit Ishikawa iteration.

Keywords: Implicit iteration; convex structure; convex metric space; fixed point
MSC 2020: 47H09, 47J25

1 Introduction

Implicit iterations provide more accurate approximation than explicit iterations from numerical point of view. Implicit iterations are used in computer programs for approximation of fixed point to reduce the computational cost of the fixed point problems. On explicit and implicit iterations in various spaces, so many works have been published [4, 8]. In computational mathematics it is of theoretical and practical importance to compare the convergence rate of iterations and to find out which iteration converges more rapidly to the fixed point. Work in this direction is done by [8, 10, 11, 13, 16, 18]. Recently some works are running in different metric spaces by [1, 9]

Zamfirescu operators [20] are most general contractive like operators which have been studied by several authors is as follows:

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a Zamfirescu operator if for each pair of points x, y in X at least one of the following is true.

(i) $d(Tx, Ty) \leq pd(x, y)$

(ii) $d(Tx, Ty) \leq q[d(x, Tx) + d(y, Ty)]$

(iii) $d(Tx, Ty) \leq r[d(x, Ty) + d(y, Tx)]$

where p, q, r are non negative constants satisfying $0 \leq p \leq 1$, $0 \leq q, r \leq \frac{1}{2}$.

Zamfirescu operators are equivalent to the following contractive contraction:

$$d(Tx, Ty) \leq c \max \left\{ d(x, y), \frac{\{d(x, Tx) + d(y, Ty)\}}{2}, \frac{\{d(x, Ty) + d(y, Tx)\}}{2} \right\} \tag{1}$$

$\forall x, y \in X, 0 < c < 1$.

Contractive condition (1) implies

$$d(Tx, Ty) \leq 2ad(x, Tx) + ad(x, y); \qquad \forall x, y \in X \tag{2}$$

where $a = \max\left\{c, \frac{c}{2-c}\right\}$.

In 1993, Rhoades [16] used the following more general contractive condition than (2) there exists $c \in [0,1)$ such that

$$d(Tx,Ty) \leq c \max\left\{d(x,y), \frac{\{d(x,Tx)+d(y,Ty)\}}{2}, d(x,Ty), d(y,Tx)\right\};\ \forall x,y \in X. \tag{3}$$

In 1995, Osilike introduced the general contractive condition, there exists $a \in (0,1)$, $L \geq 0$, such that

$$d(Tx,Ty) \leq Ld(Tx,Ty) + ad(x,y) \qquad \forall x,y \in X$$

Imoru and Olatinwo, In 2003, provides more general contractive condition
that is there exists $a \in [0.1)$ and monotone-increasing function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ s.t.

$$d(Tx,Ty) \leq \phi(d(Tx,Ty)) + ad(x,y) \qquad a \in [0,1),\ \forall x,y \in X \tag{4}$$

Definition 1.1. [2] Suppose$\{a_n\}$ and $\{b_n\}$ are two real convergent sequences with limits a and b respectively. Then $\{a_n\}$is said to converge faster than $\{b_n\}$ if

$$lim_{n\to\infty}\frac{a_n - a}{b_n - b} = 0$$

Definition 1.2. [2] Let $\{u_n\}$ and $\{v_n\}$ be two fixed point itewrations that converge to the same fixed point p on a normed space X, such that the error estimates

$$||u_n - p|| \leq a_n$$

and

$$||v_n - p|| \leq b_n$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers(converging to zero). If $\{a_n\}$converge faster than $\{b_n\}$, then we say that $\{u_n\}$ converges faster to p than $\{v_n\}$

For explicit iterations, the concept of Hybrid iteration has been defined by Khan et al. in 2013 [12]. Using that's idea, Picard S- Hybrid iteration is defined by Srivastava in [17] and Nawaz et al.in [15]. In this paper, hybrid implicit iterations for both Mann and Ishikawa have defined and prove that it will converge more rapidly than implicit Mann and Ishikawa iteration as defined in [3].

2 Implicit Hybrid Iteration

Let K be a non empty convex subset of a convex metric space X and $T : K \to K$ be a given mapping. Then for $x_0 \in K$, we define the following implicit iteration:

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n) \\ y_n &= W(z_n, Tz_n, \beta_n) \\ z_n &= W(x_n, Tx_n, \gamma_n) \end{aligned} \tag{5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

Equivalent form of iteration (5) in linear space can be written as

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n \\ y_n &= \beta_n z_n + (1-\beta_n)Tz_n \\ z_n &= \gamma_n x_n + (1-\gamma_n)Tx_n \end{aligned}$$

Putting $\gamma_n = 1$, we get implicit Ishikawa iteration(II)

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n \end{aligned} \tag{6}$$

Putting $\gamma_n = \beta_n = 1$, we get implicit Mann iteration(IM)

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n \tag{7}$$

Now we define implicit Picard Mann hybrid iteration(IPMH)

$$\begin{aligned} x_n &= Ty_n \\ y_n &= \alpha_n x_{n-1} + (1-\alpha_n)Tx_n; \qquad n \geq 1 \end{aligned} \tag{8}$$

or

$$\begin{aligned} x_n &= Ty_n \\ y_n &= W(x_{n-1}, Tx_n, \alpha_n) \end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

Implicit Picard Ishikawa hybrid iteration(IPIH)

$$\begin{aligned} x_n &= Ty_n \\ y_n &= \alpha_n x_{n-1} + (1-\alpha_n)Tz_n \\ z_n &= \beta_n x_n + (1-\beta_n)Tx_n \end{aligned} \tag{9}$$

or

$$\begin{aligned} x_n &= Ty_n \\ y_n &= W(x_{n-1}, Tz_n, \alpha_n) \\ z_n &= W(x_n, Tx_n, \beta_n) \end{aligned}$$

where $\{\alpha_n\} and \{\beta_n\}$ are sequences in $[0,1]$. As defined in above IM and II iterations $x_n, y_n \in K$ and $T : K \to K$, where K is a nonempty convex subset of convex metric space X, therefore we have $Ty_n \in K$.

Definition 2.1. [18] A map $W : X^2 \times [0,1] \to X$ is a convex structure on X if

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y) \tag{10}$$

for all $x, y, u \in X$ and $\lambda \in [0,1]$.

A metric space with convex structure W is known as convex metric space and denoted by (X, d, W)

3 Convergence results for new hybrid implicit iteration in convex metric spaces

Quasi Generalised Contractive condition: Let (X,d) be a Metric Space and $T : X \to X$ be a mapping and $p \in F(T)$, then T is said to satisfy quasi generalised contractive condition if there exists $a \in [0.1)$ and monotone-increasing function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ s.t.

$$d(Tx, Tp) \leq \phi(d(Tx,p)) + ad(x,p) \qquad a \in [0,1),\ \forall x \in X$$

Theorem 3.1. *Let K be a non empty closed convex subset of a convex metric space X and T be an operator satisfying quasi generalised contractive condition with $F(T) \neq \emptyset$. Then for $x_0 \in K$, the sequence $\{x_n\}$ defined by (8) and (9) with $\sum(1-\alpha_n) = \infty$ converges to the fixed point of T.*

Proof. Let $p \in F(T)$

$$\begin{aligned} d(x_n, p) &= d(Ty_n, p) \\ &= ad(y_n, p) \\ &= ad(W(x_{n-1}, Tx_n, \alpha_n), p) \\ &\leq a[\alpha_n d(x_{n-1}, p) + (1-\alpha_n)d(Tx_n, p)] \\ &\leq a[\alpha_n d(x_{n-1}, p) + (1-\alpha_n)ad(x_n, p)] \\ d(x_n, p) &\leq \frac{a\alpha_n}{[1-a^2(1-\alpha_n)]} d(x_{n-1}, p) \end{aligned} \tag{11}$$

If we take

$$\begin{aligned}
\frac{a\alpha_n}{1-a^2(1-\alpha_n)} &= \frac{A_n}{B_n} \\
1-\frac{A_n}{B_n} &= 1-\frac{a\alpha_n}{1-a^2(1-\alpha_n)} \\
&= \frac{1-a^2(1-\alpha_n)-a\alpha_n}{1-a^2(1-\alpha_n)} \\
&\geq 1-[a^2(1-\alpha_n)+a\alpha_n] \\
\text{or} \qquad -\frac{A_n}{B_n} &\geq -[a^2(1-\alpha_n)+a\alpha_n] \\
\text{or} \qquad \frac{A_n}{B_n} &\leq a^2(1-\alpha_n)+a\alpha_n \qquad (12)
\end{aligned}$$

that is

$$\frac{a\alpha_n}{1-a^2(1-\alpha_n)} \leq a^2(1-\alpha_n)+a\alpha_n$$

Now checking the validity of above inequality Let $x = a^2(1-\alpha_n)$ since $a \in (0,1)$ and $\alpha_n \in [0,1]$ We know that $0 \leq x \leq 1$ or $(1-x) > 0$

$$\begin{aligned}
\frac{a\alpha_n}{1-x} &\leq x + a\alpha_n \\
a\alpha_n &\leq (x+a\alpha_n)(1-x) \\
a\alpha_n &\leq x + a\alpha_n - x^2 - a\alpha_n \\
0 &\leq x - x^2 - a\alpha_n x \\
0 &\leq x(1-x-a\alpha_n)
\end{aligned}$$

Since $x \geq 0$, we have $1-x-a\alpha_n \geq 0$
Therefore we have

$$\begin{aligned}
1 &\geq a\alpha_n + x \\
1 &\geq a\alpha_n + a^2(1-\alpha_n) \\
1 &\geq a\alpha_n + a^2 - a^2\alpha_n \\
1 &\geq a^2 + a\alpha_n(1-a) \\
or, \qquad a\alpha_n(1-a) &\leq 1-a^2 \\
\alpha_n &\leq \frac{1+a}{a} \\
\alpha_n &\leq \frac{1}{a}+1
\end{aligned}$$

Since $\alpha_n \in [0,1]$ and $\frac{1}{a} > 0$ for all $a \in (0,1)$, So the inequality (12) is always true. Using inequality (12) in (11), we have

$$\begin{aligned}
d(x_n,p) &\leq [a^2(1-\alpha_n)+a\alpha_n]d(x_{n-1},p) \\
&= [a^2(1-\alpha_n)+a\alpha_n-a+a]d(x_{n-1},p) \\
&= [(1-\alpha_n)(a^2-a)+a]d(x_{n-1},p) \\
&= a[1+(1-\alpha_n)(a-1)]d(x_{n-1},p) \\
&= a[1-(1-\alpha_n)(1-a)]d(x_{n-1},p) \\
&\leq [1-(1-\alpha_n)(1-a)]d(x_{n-1},p) \\
&\leq \prod_{i=1}^{n}[1-(1-\alpha_i)(1-a)]d(x_0,p) \\
&\leq e^{-\sum_{i=1}^{n}(1-\alpha_i)(1-a)}d(x_0,p)
\end{aligned}$$

But

$$\lim_{n\to\infty}\sum_{i=1}^{n}(1-\alpha_i)=\sum_{i=1}^{\infty}(1-\alpha_i)=\infty$$

Hence

$$\lim_{n\to\infty} d(x,p)=0$$

Therefore $\{x_n\}$ converges to p.

Similarly for implicit Picard Ishikawa hybrid iteration we have

$$\begin{aligned}
d(x_n,p) &= d(Ty_n,p)\\
&\le ad(y_n,p)\\
&= ad(W(x_{n-1},Tz_n,\alpha_n),p)\\
&= a[\alpha_n d(x_{n-1},p)+(1-\alpha_n)d(Tz_n,p)]\\
&\le a[\alpha_n d(x_{n-1},p)+(1-\alpha_n)ad(z_n,p)]\\
&= a[\alpha_n d(x_{n-1},p)+(1-\alpha_n)ad(W(x_n,Tx_n,\beta_n),p)]\\
&= a[\alpha_n d(x_{n-1},p)+(1-\alpha_n)a\{\beta_n d(x_n,p)+(1-\beta_n)ad(x_n,p)\}]\\
&= a[\alpha_n d(x_{n-1},p)+a(1-\alpha_n)\beta_n d(x_n,p)+a^2(1-\alpha_n)(1-\beta_n)d(x_n,p)]\\
&= a[\alpha_n d(x_{n-1},p)+[a(1-\alpha_n)\beta_n+a^2(1-\alpha_n)(1-\beta_n)]d(x_n,p)]\\
&= a\alpha_n d(x_{n-1},p)+[a^2(1-\alpha_n)\beta_n+a^3(1-\alpha_n)(1-\beta_n)]d(x_n,p)\\
d(x_n,p) &\le \frac{a\alpha_n}{1-[a^2(1-\alpha_n)\beta_n+a^3(1-\alpha_n)(1-\beta_n)]}d(x_{n-1},p)\\
\text{or}\quad d(x_n,p) &\le \frac{a\alpha_n}{1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]}d(x_{n-1},p)
\end{aligned}$$

If we take

$$\begin{aligned}
\frac{P_n}{Q_n} &= \frac{a\alpha_n}{1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]}\\
1-\frac{P_n}{Q_n} &= 1-\frac{a\alpha_n}{1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]}\\
&= \frac{1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]-a\alpha_n}{1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]}\\
1-\frac{P_n}{Q_n} &= 1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]-a\alpha_n
\end{aligned}$$

$$\begin{aligned}
\frac{P_n}{Q_n} &\le a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]+a\alpha_n\\
&= a^2(1-\alpha_n)[-1+1+\beta_n+a(1-\beta_n)]+a\alpha_n\\
&= a^2(1-\alpha_n)[1-(1-\beta_n)+a(1-\beta_n)]+a\alpha_n\\
&= a^2(1-\alpha_n)[1+(1-\beta_n)(a-1)]+a\alpha_n\\
&= a^2(1-\alpha_n)[1-(1-\beta_n)(1-a)]+a\alpha_n\\
&\le a^2(1-\alpha_n)+a\alpha_n\\
&\le a(1-\alpha_n)+\alpha_n\\
&= a(1-\alpha_n)+\alpha_n-1+1\\
&= a(1-\alpha_n)+(\alpha_n-1)+1\\
&= a(1-\alpha_n)-(1-\alpha_n)+1\\
&= (1-\alpha_n)(a-1)+1\\
&= -(1-\alpha_n)(1-a)+1\\
&= 1-(1-\alpha_n)(1-a) \qquad (13)
\end{aligned}$$

$$\begin{aligned}\therefore \qquad d(x_n,p) &\le [1-(1-\alpha_n)(1-a)]d(x_{n-1},p)\\ &\le \prod_{i=1}^{n}[1-(1-\alpha_i)(1-a)]d(x_0,p)\\ &\le e^{-\sum_{i=1}^{n}(1-\alpha_i)(1-a)}d(x_0,p)\end{aligned}$$

But $\sum_{i=1}^{n}(1-\alpha_i)$ as $n\to\infty$ is ∞.

$$\therefore \qquad \lim_{n\to\infty} d(x_n,p)=0$$

Therefore $\{x_n\}$ converges to p. ■

4 Rate of convergence for implicit iteration

Theorem 4.1. *Let K be a non empty closed convex subset of a convex metric space X and T T be an operator satisfying quasi generalised contractive condition with $F(T)\neq\emptyset$. Then for $x_0\in K$ with $\sum(1-\alpha_n)=\infty$, the implicit Picard Mann Hybrid iteration $\{x_n\}$ defined by (8) and the implicit Picard Ishikawa hybrid iteration $\{x_n\}$ defined by (9) converges faster than implicit Mann and Ishikawa iterations respectively as defined by (7)and (6).*

Proof. For implicit Mann iteration, we have

$$\begin{aligned} d(x_n,p) &\le \alpha_n d(x_{n-1},p)+(1-\alpha_n)d(Tx_n,p)\\ &\le \alpha_n d(x_{n-1},p)+(1-\alpha_n)ad(x_n,p)\\ \implies \quad [1-a(1-\alpha_n)]d(x_n,p) &\le \alpha_n d(x_{n-1},p)\end{aligned}$$

$$\text{and so} \qquad d(x_n,p)\le \frac{\alpha_n}{[1-a(1-\alpha_n)]}d(x_{n-1},p) \tag{14}$$

For implicit Picard Mann hybrid iteration we have from above theorem

$$d(x_n,p)\le \frac{a\alpha_n}{[1-a^2(1-\alpha_n)]}d(x_{n-1},p) \tag{15}$$

$$\text{where} \qquad \frac{A_n}{B_n}=\frac{a\alpha_n}{1-a^2(1-\alpha_n)} \tag{16}$$

we have

$$\frac{a\alpha_n}{1-a^2(1-\alpha_n)}\le\frac{\alpha_n}{[1-a(1-\alpha_n)]}$$

Therefore, from Definition (1.2) and inqualities (14) and (15), we see that implicit Picard Mann hybrid iteration converges faster than implicit Mann iteration.

Similarly for implicit Ishikawa iteration, we have

$$\begin{aligned} d(x_n,p) &= d(\alpha_n x_{n-1}+(1-\alpha_n)Ty_n,p)\\ &\le \alpha_n d(x_{n-1},p)+(1-\alpha_n)d(Ty_n,p)\\ &\le \alpha_n d(x_{n-1},p)+(1-\alpha_n)ad(y_n,p)\\ &\le \alpha_n d(x_{n-1},p)+a(1-\alpha_n)d\{(\beta_n x_n+(1-\beta_n)Tx_n),p\}\\ &= \alpha_n d(x_{n-1},p)+a(1-\alpha_n)[\beta_n d(x_n,p)+a(1-\beta_n)d(x_n,p)]\\ &= \alpha_n d(x_{n-1},p)+a(1-\alpha_n)[\beta_n+a(1-\beta_n)]d(x_n,p)\end{aligned}$$

$$\text{or} \qquad d(x_n,p)\le\frac{\alpha_n}{1-a(1-\alpha_n)[\beta_n+a(1-\beta_n)]}d(x_{n-1},p) \tag{17}$$

For implicit Picard Ishikawa hybrid iteration we have from above theorem

$$d(x_n,p)\le\frac{a\alpha_n}{1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]}d(x_{n-1},p) \tag{18}$$

we have

$$\frac{a\alpha_n}{1-a^2(1-\alpha_n)[\beta_n+a(1-\beta_n)]} \leq \frac{\alpha_n}{1-a(1-\alpha_n)[\beta_n+a(1-\beta_n)]}$$

Therefore, from definition (1.2)and inequalities (17) and (18), we see that implicit Picard Ishikawa hybrid iteration converges faster than implicit Ishikawa iteration. ■

Example: Let $K=[0,1]$ and

$$T(x)=\frac{x}{4}; \qquad x\neq 0$$

and

$$\alpha_n=\beta_n=1-\frac{4}{\sqrt{n}}; \qquad n\geq 25$$

and for $n=1,2,\ldots,24$; $\alpha_n=\beta_n=0$ then for implicit Picard Mann hybrid iteration(IPMH) we have

$$x_n=Ty_n \implies x_n=\frac{y_n}{4} \implies y_n=4x_n$$

$$\begin{aligned}
y_n &= \alpha_n x_{n-1}+(1-\alpha_n)Tx_n \\
&= \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1}+\frac{4}{\sqrt{n}}.\frac{x_n}{4} \\
4x_n &= \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1}+\frac{x_n}{\sqrt{n}} \\
\left(4-\frac{1}{\sqrt{n}}\right)x_n &= \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1} \\
x_n &= \frac{\left(1-\frac{4}{\sqrt{n}}\right)}{\left(4-\frac{1}{\sqrt{n}}\right)}x_{n-1} \\
x_n &= \left(\frac{\sqrt{n}-4}{4\sqrt{n}-1}\right)x_{n-1}
\end{aligned}$$

$$\text{So,} \qquad x_n=\prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{4\sqrt{i}-1}\right)x_0 \tag{19}$$

Also for implicit Mann iteration (IM) we have

$$\begin{aligned}
x_n &= \alpha_n x_{n-1}+(1-\alpha_n)Tx_n \\
&= \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1}+\left(1-\left(1-\frac{4}{\sqrt{n}}\right)\right)\frac{x_n}{4} \\
&= \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1}+\frac{x_n}{\sqrt{n}} \\
\left(1-\frac{1}{\sqrt{n}}\right)x_n &= \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1} \\
x_n &= \left(\frac{\sqrt{n}-4}{\sqrt{n}-1}\right)x_{n-1}
\end{aligned}$$

$$x_n=\prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right)x_0 \tag{20}$$

From (19) and (20) we have

$$\begin{aligned}
\frac{x_n(IPMH)}{x_n(IM)} &= \prod_{i=25}^{n}\frac{\sqrt{i}-4}{4\sqrt{i}-1}\times\frac{\sqrt{i}-1}{\sqrt{i}-4} \\
&= \prod_{i=25}^{n}\frac{\sqrt{i}-1}{4\sqrt{i}-1}
\end{aligned}$$

with

$$0 \leq \lim_{n\to\infty} \prod_{i=25}^{n} \left(\frac{\sqrt{i}-1}{4\sqrt{i}-1} \right) \leq \lim_{n\to\infty} \prod_{i=25}^{n} \left(1 - \frac{1}{i} \right) = 0$$

Therefore

$$\lim_{n\to\infty} \left| \frac{x_n(IPIMH) - 0}{x_n(IM) - 0} \right| = 0$$

Therefore using definition (1.1), implicit Picard Mann hybrid iteration converges faster than the implicit Mann iteration to the fixed point $p = 0$.

Similarly for implicit Picard Ishikawa hybrid iteration (IPIH), we have

$$\begin{aligned}
x_n &= Ty_n \\
y_n &= \alpha_n x_{n-1} + (1-\alpha_n)Tz_n \\
z_n &= \beta_n x_n + (1-\beta_n)Tx_n \\
z_n &= \left(1 - \frac{4}{\sqrt{n}}\right) x_n + \frac{4}{\sqrt{n}} \frac{x_n}{4} \\
z_n &= \left(1 - \frac{3}{\sqrt{n}}\right) x_n \\
y_n &= \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1} + \frac{4}{\sqrt{n}} \frac{z_n}{4} \\
y_n &= \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1} + \frac{4}{\sqrt{n}} \left(1 - \frac{3}{\sqrt{n}}\right) \frac{x_n}{4} \qquad (21) \\
x_n &= Ty_n = \frac{y_n}{4} \\
\implies y_n &= 4x_n \qquad (22)
\end{aligned}$$

From (20) and (21) we have

$$\begin{aligned}
4x_n &= \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1} + \frac{1}{\sqrt{n}} \left(1 - \frac{3}{\sqrt{n}}\right) x_n \\
\left(4 - \frac{1}{\sqrt{n}} + \frac{3}{n}\right) x_n &= \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1} \\
x_n &= \frac{1 - \frac{4}{\sqrt{n}}}{4 - \frac{1}{\sqrt{n}} + \frac{3}{n}} x_{n-1} \\
&= \frac{\frac{\sqrt{n}-4}{\sqrt{n}}}{\frac{4n-\sqrt{n}+3}{n}} x_{n-1} \\
&= \frac{\sqrt{n}(\sqrt{n}-4)}{4n-\sqrt{n}+3} x_{n-1} \\
x_n &= \frac{n - 4\sqrt{n}}{4n - \sqrt{n} + 3} x_{n-1}
\end{aligned}$$

$$(x_n)_{IPIH} = \prod_{i=25}^{n} \left(\frac{i - 4\sqrt{i}}{4i - \sqrt{i} + 3} \right) x_0 \qquad (23)$$

Similarly for implicit Ishikawa iteration (II), we have

$$\begin{aligned}
x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n \\
y_n &= \beta_n x_n + (1-\beta_n)Tx_n \\
y_n &= \left(1 - \frac{4}{\sqrt{n}}\right) x_n + \frac{4}{\sqrt{n}} \frac{x_n}{4} \\
y_n &= \left(1 - \frac{3}{\sqrt{n}}\right) x_n
\end{aligned}$$

$$\text{Now} \quad x_n = \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1} + \frac{4}{\sqrt{n}}\frac{y_n}{4}$$

$$x_n = \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1} + \frac{1}{\sqrt{n}}\left(1-\frac{3}{\sqrt{n}}\right)x_n$$

$$\left(1-\frac{1}{\sqrt{n}}+\frac{3}{n}\right)x_n = \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1}$$

$$x_n = \frac{1-\frac{4}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}+\frac{3}{n}}x_{n-1}$$

$$x_n = \frac{\sqrt{n}\,(\sqrt{n}-4)}{n-\sqrt{n}+3}x_{n-1}$$

$$x_n = \frac{n-4\sqrt{n}}{n-\sqrt{n}+3}x_{n-1}$$

$$(x_n)_{II} = \prod_{i=25}^{n}\frac{i-4\sqrt{i}}{i-\sqrt{i}+3}x_0 \tag{24}$$

Therefore from (23),(24) and definition(1.1), we have

$$\frac{(x_n)_{IPIH}}{(x_n)_{II}} = \prod_{i=25}^{n}\frac{i-4\sqrt{i}}{4i-\sqrt{i}+3}\times\frac{i-\sqrt{i}+3}{i-4\sqrt{i}}$$

$$= \prod_{i=25}^{n}\frac{i-\sqrt{i}+3}{4i-\sqrt{i}+3}$$

with

$$0 \le \lim_{n\to\infty}\prod_{i=25}^{n}\left(\frac{i-\sqrt{i}+3}{4i-\sqrt{i}+3}\right) \le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right) \le 0$$

$$\Longrightarrow \left|\frac{x_n(IPIH)-0}{x_n(II)-0}\right| = 0$$

Therefore implicit Picard Ishikawa hybrid iteration converges faster than implicit Ishikawa iteration.

In the table 1 various iterations are compared and observations are listed by taking initial approximation $x_0 = 1$, $Tx = \frac{x}{4}$ and $\alpha_n = \beta_n = 1-\frac{4}{\sqrt{n}}; n \ge 25$.

The table 1 with Figures 1,2 reveals that newly introduced Picard Mann hybrid implicit iteration has better convergence rate as compared to implicit Mann iteration. Picard Ishikawa hybrid implicit iteration converges better than Picard Mann hybrid implicit iteration, Mann implicit iteration and Ishikawa implicit iteration.

Table 1: Comparison of implicit iterations for $x_0 = 1$, $Tx = \frac{x}{4}$ and $\alpha_n = \beta_n = 1-\frac{4}{\sqrt{n}}$

Number of iteration	Implicit Mann iteration	Implicit Ishikawa iteration	Implicit Picard Mann hybrid iteration	Implicit Picard Ishikawa hybrid iteration
25	0.25	0.217391	0.052631	0.0510204
26	0.0670294	0.0509705	0.00298221	0.00280581
27	0.0191074	0.0127723	0.0001803	0.00016482557
28	0.00575025	0.0033952	0.000011547	0.0000102674
29	0.00181636	0.000951573	7.7868×10^{-7}	6.741×10^{-7}
30	0.000599294	0.000279742	5.50412×10^{-8}	4.6409×10^{-8}
31	0.000205692	8.58832×10^{-5}	4.05477×10^{-9}	3.33608×10^{-9}
32	7.31828×10^{-5}	2.74322×10^{-5}	3.10632×10^{-10}	2.49457×10^{-10}
33	2.69091×10^{-5}	9.08659×10^{-6}	2.465×10^{-11}	1.93415×10^{-11}
34	1.01987×10^{-5}	3.11239×10^{-6}	2.0231×10^{-12}	1.55061×10^{-12}
35	3.97501×10^{-6}	1.09965×10^{-6}	1.7097×10^{-13}	1.28223×10^{-13}
36	1.59×10^{-6}	3.99872×10^{-6}	1.48669×10^{-14}	1.09126×10^{-14}

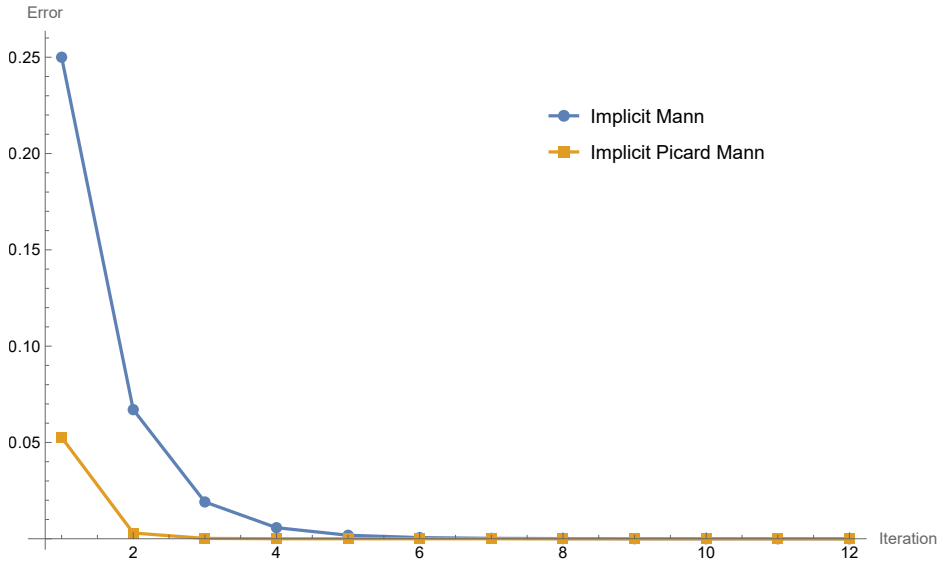


Figure 1: Implicit Mann iteration compared with implicit Picard Mann hybrid iteration

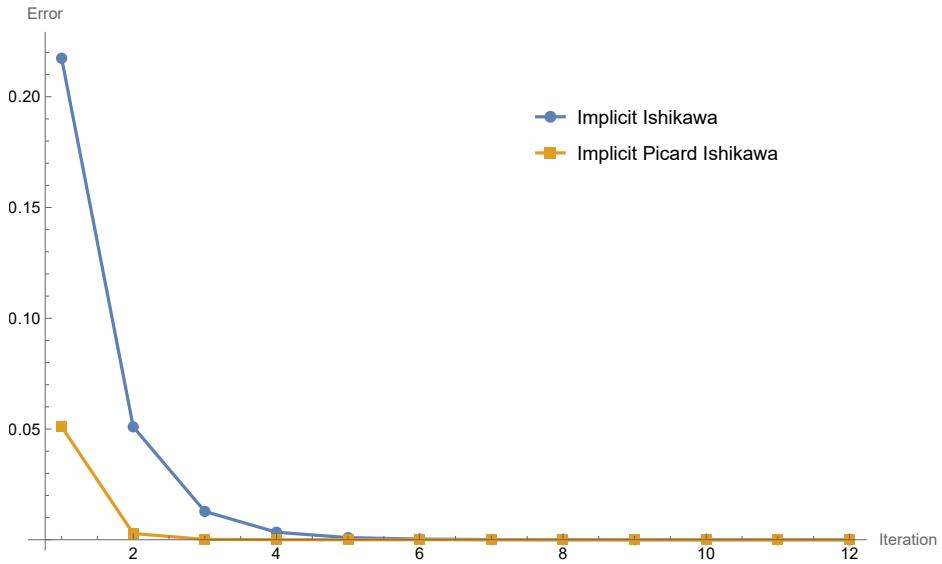


Figure 2: Implicit Ishikawa iteration compared with implicit Picard Ishikawa hybrid iteration

Data availability

All data generated or analysed during this study are included in this published article.

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Julee Srivastava: The author solely conceived, carried out, and wrote the work.

Declaration of competing interest

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References

[1] Achtoun Y, Radenovic S, Tahiri I, Sefian ML. The nonlinear contraction in probabilistic cone b-metric spaces with application to integral equation. Nonlinear Anal Model Control. 2024;29(4):658–669.

[2] Berinde V. Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators. Fixed Point Theory Appl. 2004;(2):94–105.

[3] Chug R, Malik P, Kumar V. On a new faster implicit fixed point iterative scheme in convex metric space. J Funct Spaces. 2015;2015:1–11.

[4] Ciric L, Lee BS, Rafiq A. Faster Noor iterations. Indian J Math. 2010;52(3):429–436.

[5] Ciric L, Nikolic NT. Convergence of Ishikawa iterates for multivalued mappings in convex metric spaces. Georgian Math J. 2008;15(1):39–43.

[6] Ciric L, Rafiq A, Cakic N, Ume JS. Implicit Mann fixed point iterations for pseudocontractive mapping. Appl Math Lett. 2009;22(4):581–584.

[7] Ciric L, Rafiq A, Radenovic S, Rajovic M, Ume JS. On Mann implicit iterations for strongly accretive and strongly pseudo-contractive mappings. Appl Math Comput. 2008;198(1):128–137.

[8] Ciric L, Ume JS, Khan MS. On the convergence of the Ishikawa iterates to the common fixed point of two mappings. Arch Math (Brno). 2003;39(2):123–127.

[9] Hemdanou AL, et al. Random normed K-means: a paradigm-shift in clustering within probabilistic metric spaces. arXiv:2504.039. 2025.

[10] Ishikawa S. Fixed points by new iteration method. Proc Amer Math Soc. 1974;44(1):147–150.

[11] Khan AR, Kumar V, Hussain N. Analytical and numerical treatment of Jungck-type iterative schemes. Appl Math Comput. 2014;231:521–535.

[12] Khan SH. A Picard-Mann hybrid iterative process. Fixed Point Theory Appl. 2013;2013:Article number 69.

[13] Mann WR. Mean value methods in iteration. Proc Amer Math Soc. 1953;4:506–510.

[14] Moosaei M. Fixed point theorems in convex metric spaces. Fixed Point Theory Appl. 2012;2012:Article number 164, 6 pages.

[15] Nawaz B, Ullah K, Gdawiec K. Convergence analysis of Picard-SP iteration process for generalized α-nonexpansive mappings. Numer Algorithms. 2025;98(4):1943–1964.

[16] Rhoades BE. Fixed point theorems and stability results for fixed point iteration procedure. Indian J Pure Appl Math. 1993;24(11):691–703.

[17] Srivastava J. Introduction of new Picard S-hybrid iteration with application and some results for nonexpansive mappings. Arab J Math Sci. 2022;28(1):1319–5166.

[18] Takahashi W. A convexity in metric space and nonexpansive mappings. Kodai Math Sem Rep. 1970;22:142–149.

[19] Xue Z, Zhang F. The convergence of implicit Mann iterations for week generalized ϕ-hemicontractive mappings in real Banach spaces. J Inequal Appl. 2013;2013:1–11.

[20] Zamfirescu T. Fixed point theorems in metric spaces. Arch Math (Basel). 1972;23:292–298.